Analytic, geometric and measured aspects of lamplighter-like groups

Under the supervision of Anthony Genevois and Romain Tessera

Vincent Dumoncel

PhD defense

July 7, 2026

Coarse geometry

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What if we look at a metric space from far away?

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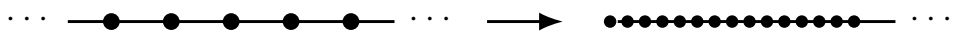

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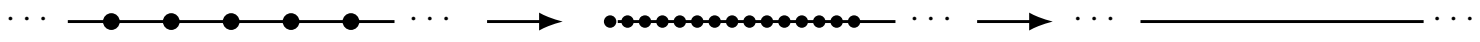

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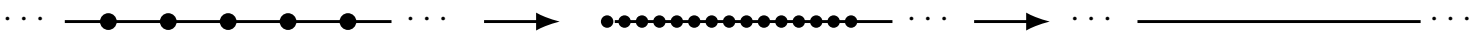

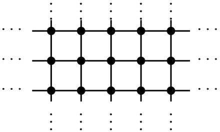

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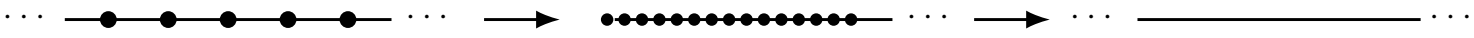

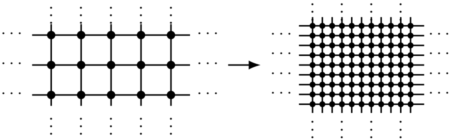

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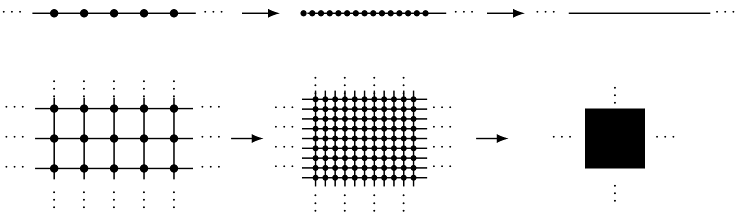

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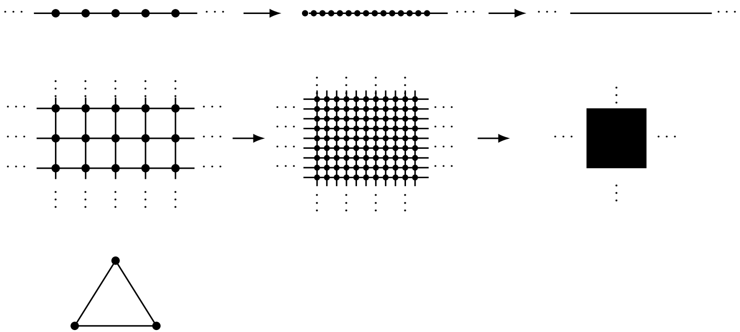

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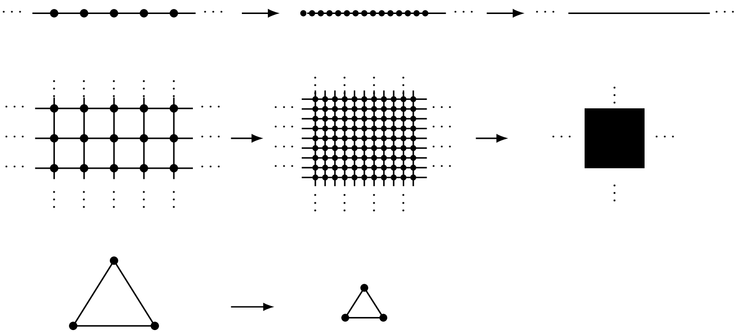

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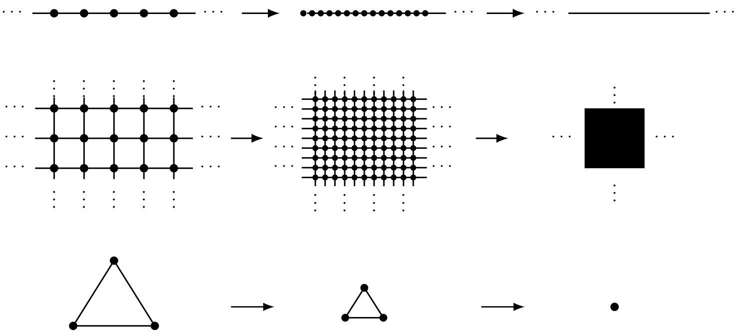

biLipschitz maps

Definition

A map $f : X \to Y$ is a *biLipschitz equivalence* if it is surjective and there is $C > 0$ such that

$$\frac{1}{C} \cdot d_X(x,y) \leq d_Y(f(x), f(y)) \leq C \cdot d_X(x,y),\ x,y \in X.$$

Quasi-isometries: definition

Definition

Let X, Y be metric spaces. A map $f : X \to Y$ is a (C, K)*−quasi-isometry* if there are $C \geq 1$, $K \geq 0$ such that

$$\forall x, y \in X, \frac{1}{C} \cdot d_X(x, y) - K \leq d_Y(f(x), f(y)) \leq C \cdot d_X(x, y) + K$$

and $d_Y(y, f(X)) \leq K$.

If such a map exists, we say X and Y are *quasi-isometric*, denoted $X \sim_{Q.I.} Y$.

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- $\mathbb{Z} \sim_{Q.I.} \mathbb{R}$ and $\mathbb{Z}^2 \sim_{Q.I.} \mathbb{R}^2$;
- Bounded metric spaces are quasi-isometric to $\{*\}$.

Contents

Coarse geometry and group theory

Introduction: where is coarse geometry in group theory?

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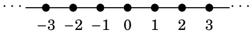

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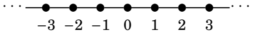

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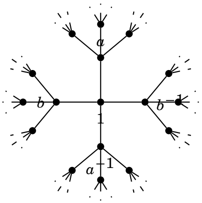

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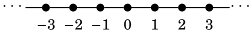

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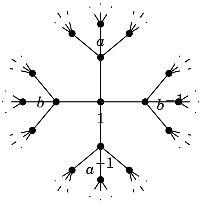

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- *Word length of $g \in G$*:

$$|g|_S := \min \left\{ n \geq 0 : \exists s_1, \ldots, s_n \in S \cup S^{-1}, g = s_1 \ldots s_n \right\}.$$

- *Word metric*:

$$d_S(g, h) := |g^{-1}h|_S, \; g, h \in G.$$

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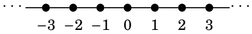

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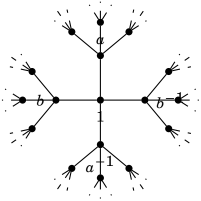

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- *Word metric*:

$$d_S(g, h) := |g^{-1}h|_S, \ g, h \in G.$$

- If S, S' are different generating sets of G, then (G, d_S) and $(G, d_{S'})$ are biLipschitz equivalent.

Introduction: growth of groups

Gromov's program. Classify, or at least "understand", finitely generated groups up to quasi-isometry.

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The *growth function* $\gamma_G : \mathbb{N} \to \mathbb{N}$ of G, defined as

$$\gamma(n) = |B_{d_S}(1_G, n)|$$

is a quasi-isometry invariant, up to $\simeq$, where

- $f \preccurlyeq g$ if there is $C > 0$, $f(n) \leq Cg(Cn)$ for n large enough;
- $f \simeq g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

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$\gamma_{\mathbb{Z}}(n) \simeq n$ and $\gamma_{F_2}(n) \simeq e^n$, so $\mathbb{Z} \nsim_{Q.I.} F_2$. Likewise, $\mathbb{Z}^d \nsim_{Q.I.} \mathbb{Z}^{d'}$ if $d \neq d'$.

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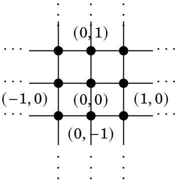

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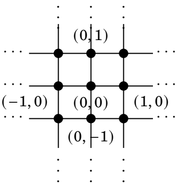

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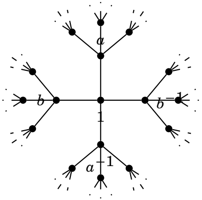

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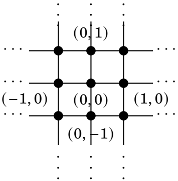

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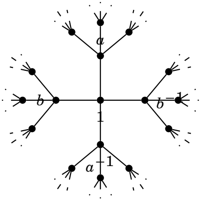

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Theorem (Milnor, Wolf, 1968)

A finitely generated solvable group of subexponential growth is virtually nilpotent.

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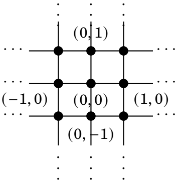

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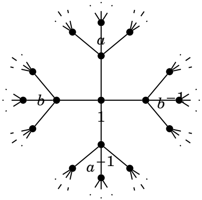

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Theorem (Milnor, Wolf, 1968)

A finitely generated solvable group of subexponential growth is virtually nilpotent.

Theorem (Gromov, 1981)

A finitely generated group has polynomial growth if and only if it is virtually nilpotent.

Can algebraic properties be in fact geometric properties?

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Being virtually solvable

Examples: $\mathrm{BS}(1, n)$, $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$, $n \geq 2$
No (Dyubina 2000), using lamplighters

Being virtually polycyclic

Examples: $\mathrm{SOL}(\mathbb{Z})$, finitely generated solvable subgroups of $\mathrm{GL}(n, \mathbb{Z})$ (Malcev, 1951)
Open question, try your luck

Being virtually nilpotent

Examples: $\mathrm{Heis}(\mathbb{Z})$, $\mathrm{UT}(n, \mathbb{Z})$, ...
Yes, this is exactly Gromov's theorem

Being virtually abelian

Examples: $\mathbb{Z}^n$ and its finite extensions
Yes, using Gromov's theorem

Definition of wreath products

Given G, H two groups, their *wreath product* $G \wr H$ is the group

$$G \wr H := \left(\bigoplus_H G \right) \rtimes H$$

where H acts on the direct sum by shifting the coordinates:

$$h' \cdot (g_h)_{h \in H} := (g_{h'^{-1}h})_{h \in H}.$$

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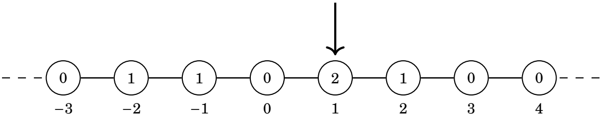

An element of $\mathbb{Z} \wr \mathbb{Z}$.

For $(c, p) \in G \wr H$:

- $c \colon H \longrightarrow G$ is a finitely supported colouring of H;
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Description of lamplighters

If $G = \langle A \rangle$ and $H = \langle B \rangle$ are finitely generated, then

$$C = \{(\delta_a, 1_H) : a \in A\} \cup \{(\mathbf{1}, b) : b \in B\}$$

is finite and generates $G \wr H$ (where δ_a sends 1_H to a and any other $h \neq 1_H$ to 1_G).

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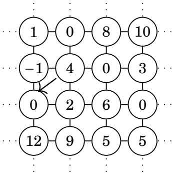

A path of length two in the Cayley graph of $\mathbb{Z} \wr \mathbb{Z}^2$.

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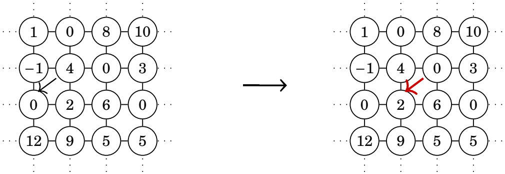

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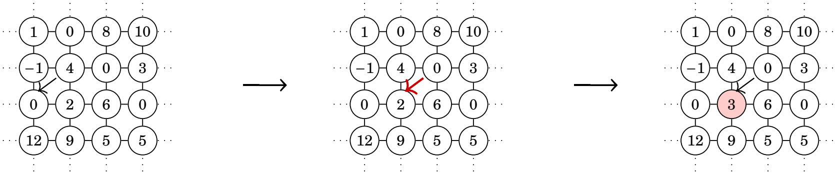

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- either we only move the arrow to a neighbouring vertex in H, and the coloring stays the same;
- or the arrow stays on the vertex where it stands, but changes the color of this vertex, and replaces it with an adjacent color in $\mathrm{Cay}(G, A)$.

QI rigidity of lamplighters

Question

When are $G_1 \wr H_1$ and $G_2 \wr H_2$ quasi-isometric?

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Proposition (Dyubina, 2000)

Let A, B and C be finitely generated groups. If A and B are biLipschitz equivalent, then so are $A \wr C$ and $B \wr C$.

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Later, Eskin, Fisher and Whyte proved a complete classification of lamplighters over $\mathbb{Z}$:

Theorem (Eskin-Fisher-Whyte, 2013)

Let $n, m \geq 2$. Then $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z}$ are quasi-isometric if and only if n and m are powers of a common number.

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Even more recently, Genevois and Tessera proved a similar criterion for lamplighters based on one-ended groups:

QI rigidity of lamplighters

Theorem (Genevois-Tessera, 2024)

Let $n, m \geq 2$. Let H_1, H_2 be finitely presented groups. Assume that H_1 is one-ended. Then:

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Let $n, m \geq 2$. Let H_1, H_2 be finitely presented groups. Assume that H_1 is one-ended. Then:

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- If H_1 is amenable, then $\mathbb{Z}/n\mathbb{Z} \wr H_1$ and $\mathbb{Z}/m\mathbb{Z} \wr H_2$ are quasi-isometric if and only if there are integers $a, r, s \geq 1$ such that $n = a^r$, $m = a^s$ and there is a **quasi-$\frac{s}{r}$-to-one** quasi-isometry $H_1 \to H_2$.

where a quasi-isometry $f : G \to H$ is **quasi-k-to-one** if there is $C > 0$

$$\big| k|A| - |f^{-1}(A)| \big| \leq C \cdot |\partial_H A|$$

for any $A \subset H$ finite. The notion:

- "quantifies" how far from a bijection f is (in relation with Whyte's theorem);
- is only relevant between amenable groups.

Quasi-isometric rigidity of permutational lamplighters

QI rigidity of permutational lamplighters

Given G, H and an action of H on a set X, the *permutational wreath product* $G \wr_X H$ is defined as

$$G \wr_X H = \Big(\bigoplus_X G \Big) \rtimes H$$

where H acts on the direct sum through its initial action on X:

$$h \cdot (g_x)_{x \in X} := (g_{h^{-1} \cdot x})_{x \in X}.$$

Description of permutational lamplighters

Picture to keep in mind when $X = H/N$:

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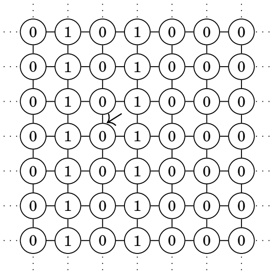

An element $(c, p) \in \mathbb{Z}/2\mathbb{Z} \wr_{\mathbb{Z}} \mathbb{Z}^2$ is a colouring of $\mathbb{Z}-$cosets in $\mathbb{Z}^2$ with colors 0 or 1, and an arrow pointing at $p \in \mathbb{Z}^2$.

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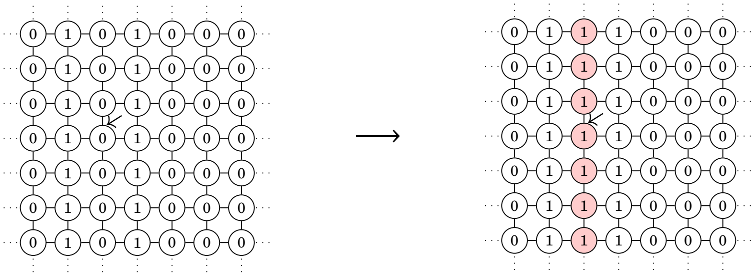

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Statement of the result

Theorem (D., 2026+)

Let E, F be non-trivial finite groups. Let G, H be finitely presented groups with normal infinite finitely generated subgroups of infinite index $M \lhd G$, $N \lhd H$.

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Let E, F be non-trivial finite groups. Let G, H be finitely presented groups with normal infinite finitely generated subgroups of infinite index $M \lhd G$, $N \lhd H$.

- If M is not co-amenable in G, then $E \wr_{G/M} G$ and $F \wr_{H/N} H$ are quasi-isometric if and only if $|E|$ and $|F|$ have the same prime divisors and there exists a **quasi-isometry of pairs** $(G, M) \to (H, N)$.

where a **quasi-isometry of pairs** $(G, M) \to (H, N)$ is a map sending any $M-$coset at a uniform bounded distance from an $N-$coset and having a quasi-inverse doing the same thing for $N-$cosets.

Statement of the result

Theorem (D., 2026+)

Let E, F be non-trivial finite groups. Let G, H be finitely presented groups with normal infinite finitely generated subgroups of infinite index $M \lhd G$, $N \lhd H$.

- If M is not co-amenable in G, then $E \wr_{G/M} G$ and $F \wr_{H/N} H$ are quasi-isometric if and only if $|E|$ and $|F|$ have the same prime divisors and there exists a **quasi-isometry of pairs** $(G, M) \to (H, N)$.
- If M is co-amenable in G, then $E \wr_{G/M} G$ and $F \wr_{H/N} H$ are quasi-isometric if and only if there exist $a, r, s \geq 1$ such that $|E| = a^r$, $|F| = a^s$ and a quasi-isometry of pairs $(G, M) \to (H, N)$ inducing a quasi-$\frac{s}{r}$-to-one quasi-isometry $G/M \to H/N$.

where a **quasi-isometry of pairs** $(G, M) \to (H, N)$ is a map sending any $M-$coset at a uniform bounded distance from an $N-$coset and having a quasi-inverse doing the same thing for $N-$cosets.

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Theorem (D., 2026+)

Let E, F be non-trivial finite groups. Let G, H be finitely presented groups with normal infinite finitely generated subgroups of infinite index $M \triangleleft G$, $N \triangleleft H$. Assume that G (resp. H) is not coarsely separable by any collection of subspaces that uniformly quasi-isometrically embed into M (resp. N).

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Step 1: a quasi-isometry between permutational lamplighters must preserve cosets of the base groups.

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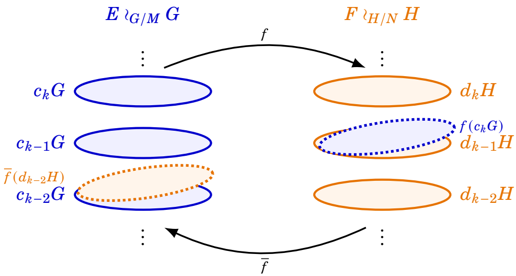

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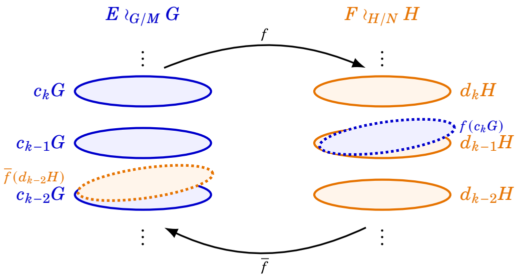

Hence, under our assumptions, any quasi-isometry f between $E \wr_{G/M} G$ and $F \wr_{H/N} H$ is a quasi-isometry of pairs

$$f\colon (E \wr_{G/M} G, G) \to (F \wr_{H/N} H, H).$$

Cone-offs of graphs

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Definition

Let X be a graph together with a partition $\mathcal{C}$ of its vertex set $V(X)$. The *cone-off* of X with respect to $\mathcal{C}$ is the graph $\mathrm{CO}(X, \mathcal{C})$ obtained from X by adding an edge between two vertices of $V(X)$ whenever they belong to a common piece $C \in \mathcal{C}$.

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Main example for us:

$$\mathrm{CO}(E \wr_{G/M} G, \{M\text{-cosets}\}) \sim_{Q.I.} E \wr G/M.$$

A brief sketch of proof

$$
\begin{tikzcd}
\left(E \wr_{G/M} G, \{M\text{-cosets}\}\right) \arrow[r, "f"] \arrow[d, "CO"'] & \left(F \wr_{H/N} H, \{N\text{-cosets}\}\right) \arrow[d, "CO"] \\
CO\left(E \wr_{G/M} G, \{M\text{-cosets}\}\right) \arrow[r, "f^{\mathrm{in}}"] \arrow[d] & CO\left(F \wr_{H/N} H, \{N\text{-cosets}\}\right) \arrow[d] \\
E \wr G/M \arrow[r, "f^{\mathrm{in}}"'] & F \wr H/N
\end{tikzcd}
$$

Now: M co-amenable in $G \Longleftrightarrow G/M$ amenable, so the conclusion follows from Genevois-Tessera's theorem.

Towards lampshufflers and isoperimetric profiles

Definition of lampshufflers

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QI rigidity of lampshufflers

At the level of quasi-isometric rigidity, we have:

Theorem (Genevois-Tessera, 2024)

Let H_1, H_2 be finitely presented one-ended groups. Then $\mathsf{Shuffler}(H_1)$ and $\mathsf{Shuffler}(H_2)$ are quasi-isometric if and only if there is a biLipschitz equivalence $H_1 \to H_2$.

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as well as:

Theorem (Correia-D., 2026+)

Let $n, m \geq 1$ and $d, k \geq 1$. Then $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^d)$ and $\mathsf{Shuffler}^{\circ m}(\mathbb{Z}^k)$ are quasi-isometric if and only if $n = m$ and $d = k$.

where *iterated lampshufflers* are defined inductively by

$$\mathsf{Shuffler}^{\circ 0}(H) := H,\ \mathsf{Shuffler}^{\circ (n+1)}(H) := \mathsf{Shuffler}(\mathsf{Shuffler}^{\circ n}(H)).$$

Isoperimetric profiles

Definition

Let $G = \langle S_G \rangle$ be a finitely generated group. Its *isoperimetric profile* is the function $j_{1,G} : \mathbb{N} \to \mathbb{R}_+$ defined as

$$j_{1,G}(n) := \sup_{A \subset G, |A| \leq n} \frac{|A|}{|\partial_G A|}$$

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Let F be a non-trivial finite group. Let H be a finitely generated amenable group whose isoperimetric profile satisfies:

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Up to now, Assumption $(*)$ is satisfied by any finitely generated amenable group for which we know the isoperimetric profile.

Isoperimetric profiles of lampshufflers

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and this bound is optimal when H has polynomial growth:

$$j_{1,\mathsf{Shuffler}(H)}^{-1}(n) \simeq (n^d)^{n^d}$$

where $d = \deg(H) \geq 1$.

Isoperimetric profiles of lampshufflers

Hence:

$$j_{1,\mathsf{Shuffler}(H)}(n) \simeq \left(\frac{\ln(n)}{\ln(\ln(n))}\right)^{\frac{1}{d}}$$

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Problem: upper bound far from being optimal, because it only depends on the growth of H. Examples: $j_{1,\mathsf{Shuffler}(F \wr \mathbb{Z})}$ and $j_{1,\mathsf{Shuffler}(F \wr (F \wr \mathbb{Z}))}$, F finite and $\neq \{1\}$.

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- For the lower bound: given $(F_n)_n$ a Følner sequence of H, $(L_n) = ((\mathsf{FSym}(F_n), F_n))_n$ is a Følner sequence of $\mathsf{Shuffler}(H)$;

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- For the upper bound: $j_{1,H}(\ln(n))$ is the profile of a lamplighter $F \wr H$, so find lamplighter substructures inside $\mathsf{Shuffler}(H)$ and use monotonicity of the profile (Erschler 2003).

Isoperimetric profiles of lampshufflers

Corollary (Correia-D., 2026+)

Let H be a finitely generated amenable group whose isoperimetric profile $j_{p,H}$ satisfies Assumption $(*)$. Assume moreover that

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For instance, $j_{1,\mathsf{Shuffler}(F\wr\mathbb{Z})}(n) \simeq \ln(\ln(n))$ whereas, as expected, the corollary provides

$$j_{1,\mathsf{Shuffler}(F\wr(F\wr\mathbb{Z}))}(n) \simeq \ln(\ln(\ln(n))).$$

Isoperimetric profiles of iterated lampshufflers

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- If H has polynomial growth of degree $d \geq 1$, then

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Now:

"vague" question

How to compare large-scale geometries of two lampshufflers when they are not quasi-isometric?

Quantitative orbit equivalence...

Orbit equivalence

Orbit equivalence

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Two groups G and H are *orbit equivalent* if they both act freely and pmply on a standard probability space (X, μ) with the same orbits: $G \cdot x = H \cdot x$ for $\mu-$almost every $x \in X$.

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Theorem (Ornstein-Weiss, 1980)

Any two infinite amenable groups are orbit equivalent.

Quantitative orbit equivalence

To get a more rigid notion, one can *quantify* the cocycles.

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Indeed, a coupling gives rise to two maps $c_{G,H} : G \times X \to H$, $c_{H,G} : H \times X \to G$ uniquely defined by

$$c_{G,H}(g,x) \cdot x = g \cdot x,\ c_{H,G}(h,x) \cdot x = h \cdot x$$

for all $g, h \in H$ and almost every $x \in X$.

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$$G = \mathbb{Z}^2 = \langle e_1, e_2 \rangle \qquad\qquad H = \mathbb{Z} = \langle 1 \rangle$$

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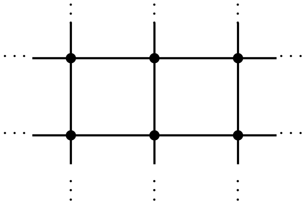

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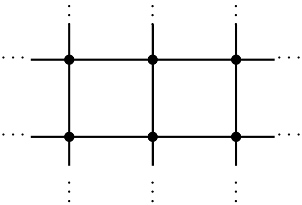

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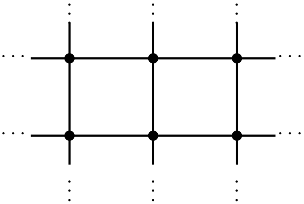

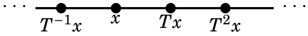

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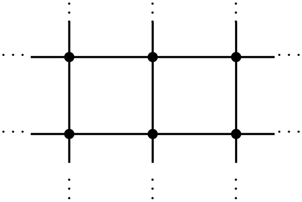

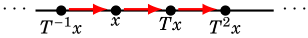

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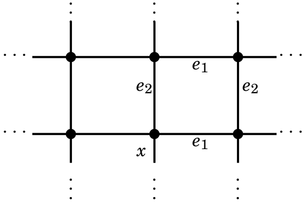

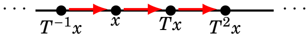

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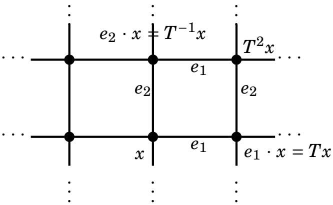

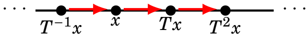

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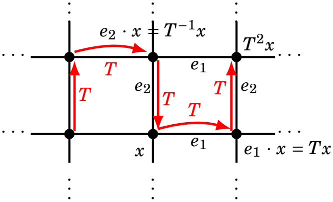

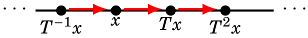

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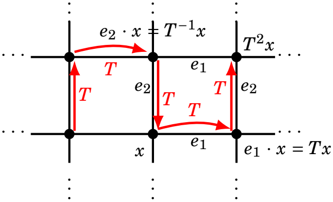

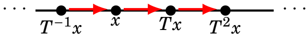

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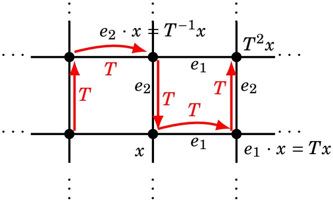

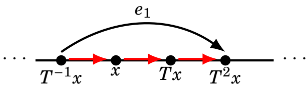

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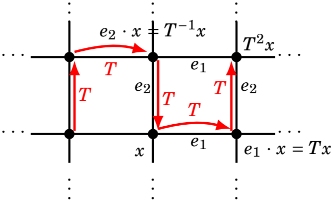

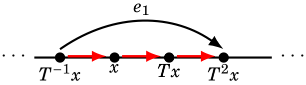

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$$c_{\mathbb{Z}^2,\mathbb{Z}}(e_1, T^{-1}x) = 3$$

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Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps. The cocycle $c_{G,H}$ is $\varphi-$*integrable* if for any $g \in G$, there is $C_g > 0$ such that

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- A cocycle is L^p if it is $\varphi-$integrable with $\varphi(x) = x^p$, $p > 0$;
- L^0: no requirement;
- A $(\varphi,\varphi)-$integrable coupling is simply called $\varphi-$integrable;
- If $\varphi \simeq \varphi'$, then $c_{G,H}$ is $\varphi-$integrable if and only if it is $\varphi'-$integrable.

Interplay between quasi-isometries and orbit equivalence

Geometry

Measure theory

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Geometry **Measure theory**

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$$\begin{array}{ccc}
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BiLipschitz equivalence $\Downarrow$ Quasi-isometry

L^∞–OE $\Downarrow$ (Furman 1999) L^∞–ME mutually cobounded

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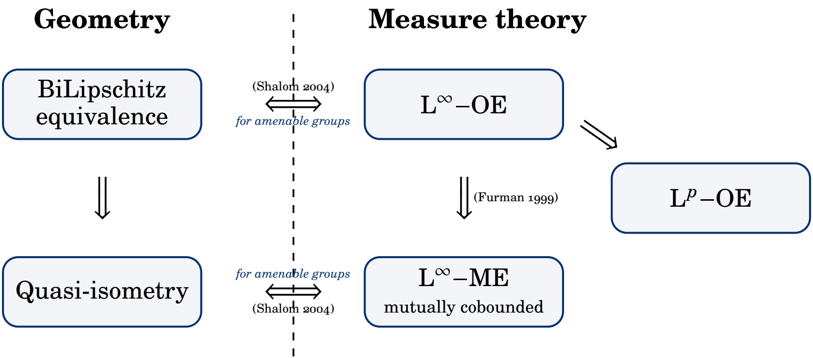

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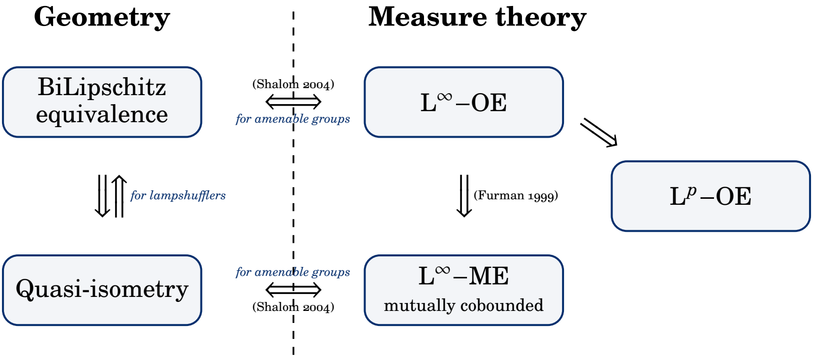

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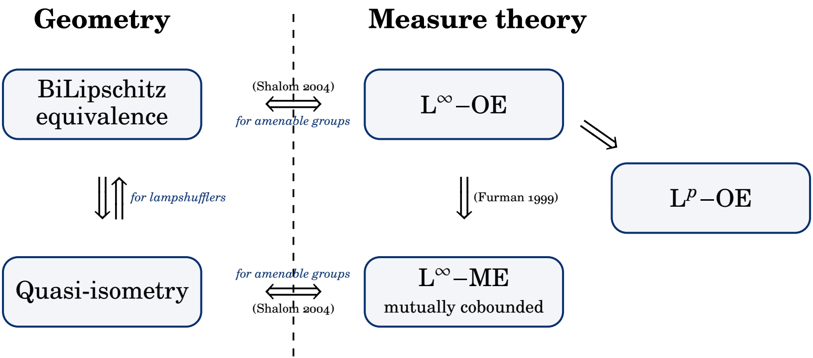

More precise question

If two lampshufflers are not quasi-isometric, can they still be L^p−OE for some values of $p > 0$?

Relation of QOE with isoperimetric profiles

Theorem (Delabie, Koivisto, Le Maître, Tessera, 2023)

Let G and H be finitely generated groups. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be increasing such that $t \mapsto \frac{t}{\varphi(t)}$ is increasing. If there exists a $(\varphi, \mathrm{L}^0)-$orbit equivalence coupling from G to H, then

$$\varphi \circ j_{1,H}(n) \preccurlyeq j_{1,G}(n).$$

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In practice: $j_{1,G}(Cx) = O(j_{1,G}(x))$ for $C > 0$ (Assumption $(*)$), so the asymptotic inequality of the theorem becomes

$$\varphi \preccurlyeq j_{1,G} \circ j_{1,H}^{-1},$$

i.e. an explicit upper bound on the integrability of the cocycle $c_{G,H}$ when there is an OE from G to H.

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Hence for instance:

Corollary (Delabie, Koivisto, Le Maître, Tessera, 2023, Correia, 2025)

Let $k, \ell \geq 1$. There is an (L^p, L^0)−orbit equivalence coupling from $\mathbb{Z}^{k+\ell}$ to $\mathbb{Z}^k$ if and only if $p < \frac{k}{k+\ell}$.

...between lampshufflers

Couplings between lampshufflers: stability result

Goal: find optimal orbit equivalence couplings between lampshufflers.

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First, a stability property for lamplighters:

Theorem (Delabie, Koivisto, Le Maître, Tessera, 2023)

Let F, G, H be finitely generated groups and let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing maps. If G and H are $(\varphi, \psi)-$orbit equivalent, then $F \wr G$ and $F \wr H$ are $(\varphi, \psi)-$orbit equivalent.

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We prove a similar stability result for lampshufflers:

Theorem (Correia-D., 2026+)

Let H and K be two finitely generated groups. Let $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing maps. If there is a $(\varphi, \psi)-$integrable orbit equivalence coupling from H to K, then there is a $(\varphi, \psi)-$integrable orbit equivalence coupling from $\mathsf{Shuffler}(H)$ to $\mathsf{Shuffler}(K)$.

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Corollary (Correia-D., 2026+)

If H and K are $(\varphi, \psi)-$orbit equivalent, then so are $\mathsf{Shuffler}^{\circ n}(H)$ and $\mathsf{Shuffler}^{\circ n}(K)$, for any $n \geq 0$.

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For optimality, we use our computations of profiles of lampshufflers (given the upper bound "$\varphi \preccurlyeq j_{1,H} \circ j_{1,G}^{-1}$").

Optimal couplings between iterated lampshufflers

Theorem (Correia-D., 2026+)

Let $k, \ell, n \geq 1$. Let $p \geq 0$. Then there exists an $(\mathrm{L}^p, \mathrm{L}^0)$−orbit equivalence from $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^{k+\ell})$ to $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^k)$ if and only if $p < \frac{k}{k+\ell}$.

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whereas:

Theorem (Correia-D., 2026+)

Let $k, n \geq 1$ and $\ell \geq 0$. Then $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^{k+\ell})$ and $\mathsf{Shuffler}^{\circ n}(\mathbb{Z}^k)$ are quasi-isometric if and only if $\ell = 0$.

Open questions

- Does there exist a quasi-isometric (or coarse) embedding of $\mathsf{Shuffler}(\mathbb{Z})$ in $\mathbb{Z} \wr \mathbb{Z}$?

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- For which values of $p > 0$ can $\mathsf{Shuffler}(\mathbb{Z})$ and $\mathbb{Z} \wr \mathbb{Z}$ be L^p−orbit equivalent?

Thank you!